

UNIVERSAL AND HOMOGENEOUS STRUCTURES ON THE URYSOHN AND GURARIJ SPACES

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ABSTRACT. Using Fraïssé theoretic methods we enrich the Urysohn universal space by universal and homogeneous closed relations, retractions, closed subsets of the product of the Urysohn space itself and some fixed compact metric space, L -Lipschitz map to a fixed Polish metric space. The latter lifts to a universal linear operator of norm L on the Lipschitz-free space of the Urysohn space.

Moreover, we enrich the Gurarij space by a universal and homogeneous closed subspace and norm one projection onto a 1-complemented subspace. We construct the Gurarij space by the classical Fraïssé theoretic approach.

INTRODUCTION

The Urysohn space and the Gurarij space are two main examples of universal and homogeneous metric structures. The former, in the category of separable complete metric spaces, was constructed by P. Urysohn already in 1927 in [12], while the latter, in the category of separable Banach spaces, was constructed by Gurarij in 1966 in [5]. The characterizing property of the Urysohn space \mathbb{U} is that for any finite subset M (including an empty subset) of \mathbb{U} , any ‘abstract’ finite metric extension M' of M can be realized in \mathbb{U} . This property also implies that \mathbb{U} contains an isometric copy of any separable metric space. The characterizing property of the Gurarij space is that for any finite dimensional subspace X of \mathbb{G} , any ‘abstract’ finite dimensional extension X' can be realized with ‘ ε -accuracy’ for any $\varepsilon > 0$. This property also implies that \mathbb{G} contains a linearly isometric copy of any separable normed vector space.

The aim of this paper is to enrich the Urysohn and Gurarij spaces by some additional structure so that they remain universal and homogeneous with that added structure. We refer the reader to [3] where

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the Gurarij space was enriched by a universal and homogeneous linear operator for an example of a result of that kind.

We roughly summarize the main results below. The precise statements are in the appropriate sections.

Theorem 0.1. *The Urysohn universal metric space can be enriched:*

- *with finitely many universal and homogeneous closed relations of an arbitrary arity,*
- *with a universal and homogeneous 1-Lipschitz retraction onto a universal and homogeneous retract subspace,*
- *with a universal and homogeneous closed subset of itself and an arbitrary fixed compact metric space,*
- *with a universal and homogeneous L -Lipschitz function to any fixed Polish metric space, for any fixed $L > 0$; moreover, when the fixed Polish metric space is in fact a Banach space, then the L -Lipschitz functions lift to a universal linear operator of norm L from the Holmes space to that fixed Banach space.*

Theorem 0.2. *The Gurarij universal Banach space can be enriched:*

- *with a universal and homogeneous closed subspace,*
- *with a universal and homogeneous norm one projection onto a universal and homogeneous 1-complemented subspace.*

1. PRELIMINARIES

We expect the reader to be familiar with basic Fraïssé theoretic constructions. At least knowing the standard Fraïssé theoretic construction of the Urysohn space is important for understanding of our construction (in contrast to the other popular construction due to Katětov in [8]). We refer the reader to Chapter 5 of the book [11] which is devoted to the Urysohn space and contains a construction of this space which is in the same spirit as our constructions here. For a general exposition of Fraïssé theory, we refer the reader to Chapter 7 of [6] which is devoted to this topic, and then to [9] that contains a general category-theoretic approach to Fraïssé theory.

We make a brief overview of Fraïssé theory here. Let \mathcal{K} be a class of (finitely generated) structures of some type and moreover, let \mathcal{E} be some class of distinguished embeddings between structures of \mathcal{K} . We say that $(\mathcal{K}, \mathcal{E})$

- is countable if it contains only countably many structures up to isomorphism from \mathcal{E} ,

- has the joint-embedding property if for every $A, B \in \mathcal{K}$ there is $C \in \mathcal{K}$ and embeddings of A , resp. B , into C that belong to \mathcal{E} ,
- has the amalgamation property if for every A, B, C and embeddings $\iota_B : A \hookrightarrow B$ and $\iota_C : A \hookrightarrow C$ from \mathcal{E} there exist $D \in \mathcal{K}$ and embeddings $\rho_B : B \hookrightarrow D$ and $\rho_C : C \hookrightarrow D$ from \mathcal{E} such that $\rho_C \circ \iota_C = \rho_B \circ \iota_B$.

If $(\mathcal{K}, \mathcal{E})$ satisfies all these conditions then we call it a Fraïssé class. The following is the classical Fraïssé theorem. Let us note that when $A \in \mathcal{K}$ and K is a direct limit $K_1 \rightarrow K_2 \rightarrow \dots$ of structures from \mathcal{K} , then by saying that an embedding $\iota : A \hookrightarrow K$ belongs to \mathcal{E} we mean that there exists n so that $\iota \in \mathcal{E}$ in fact goes from A to K_n .

Theorem 1.1 (Fraïssé theorem). *Let $(\mathcal{K}, \mathcal{E})$ be a Fraïssé class. Then there exists a limit structure K called the Fraïssé limit direct limit $K_1 \rightarrow K_2 \rightarrow \dots$, where $K_i \in \mathcal{K}$, for every i , and the embedding between K_i and K_{i+1} is in \mathcal{E} , for every i , with the following properties that characterize it up to isomorphism (among countable, resp. separable, structures of the same type):*

- for every $A \in \mathcal{K}$ there exists an embedding $\iota_A : A \hookrightarrow K$ from \mathcal{E} ,
- for every $A, B \in \mathcal{K}$ and embeddings $\iota_A : A \hookrightarrow K$ and $\rho : A \hookrightarrow B$ from \mathcal{E} there exists an embedding $\iota_B : B \hookrightarrow K$ from \mathcal{E} such that $\iota_B \circ \rho = \iota_A$.

One can derive two additional properties of K from those two stated above:

- (1) If L is a structure that is a direct limit $L_1 \rightarrow L_2 \rightarrow \dots$, where for each i , $L_i \in \mathcal{K}$ and the embedding from L_i to L_{i+1} is from \mathcal{E} , then there exists an embedding of L into K .
- (2) If $A, B \in \mathcal{K}$ are isomorphic and embedded into K via $\iota_A : A \hookrightarrow K$ and $\iota_B : B \hookrightarrow K$ from \mathcal{E} , then there exists an automorphism of K that sends $\iota_A[A]$ onto $\iota_B[B]$.

The first property is called the *universality* of K ; the latter is called the *homogeneity*, or sometimes ultrahomogeneity, of K .

The best example for us is the following. Let \mathcal{K} be the class of all finite metric spaces with rational distances and consider the class \mathcal{E} of all isometric embeddings between them. Then $(\mathcal{K}, \mathcal{E})$ is a Fraïssé class with the limit being the *rational Urysohn space* \mathbb{QU} . We get from Theorem 1.1 the following characterization of \mathbb{QU} .

Fact 1.2. *\mathbb{QU} is the unique countable rational metric space with the property that for every finite subspace A and its finite extension, which is still rational, this extension of A is actually realized within \mathbb{QU} .*

The property from Fact 1.2 is called rational finite-extension property, or in the case when this extension is just by one point, a *rational one-point extension property*. Obviously, one-point extension property implies finite-extension property.

The metric completion of \mathbb{QU} is the Urysohn space \mathbb{U} . We again refer to Chapter 5 of [11] where this is proved.

In the section on the Gurarij space, we shall present a new construction of the Gurarij space that is Fraïssé theoretic in this classical sense. That will help us use the similar ideas and techniques that we use for enriching the structure on the Urysohn space to enrich the Gurarij space by additional structure as well.

Since we will amalgamate metric structures often in the rest of the paper we make a brief description of that procedure here.

Definition 1.3 ((Greatest) metric amalgamation). Suppose we are given metric spaces X_0, X_1, X_2 such that X_0 is a subspace of both X_1 and X_2 . The underlying set of the amalgam X_3 will be the disjoint union $X_0 \amalg (X_1 \setminus X_0) \amalg (X_2 \setminus X_0)$. We need to define the distance between x and y , where $x \in X_1 \setminus X_0$ and $y \in X_2 \setminus X_0$. In order to satisfy the triangle inequalities, we need to have

$$\sup_{z \in X_0} |d_{X_1}(x, z) - d_{X_2}(z, y)| \leq d(x, y) \leq \inf_{z \in X_0} d_{X_1}(x, z) + d_{X_2}(z, y).$$

If we take for all such pairs the greatest extreme then it is straightforward to check that this defines a metric on X_3 extending those on X_1 and X_2 , and that such a metric is the greatest possible. If $X_1 \setminus X_0 = \{x\}$ and $X_2 \setminus X_0 = \{y\}$ then we can of course define the distance between x and y to be any number in between these two extremes.

Let us also note that whenever we consider some metric on a product of metric spaces we mean the *sum metric*.

2. THE URYSOHN SPACE

Theorem 2.1. *Let $n_1 \leq \dots \leq n_m$ be an arbitrary non-decreasing sequence of natural numbers. Then there exist closed relations (subsets) $F_{n_i} \subseteq \mathbb{U}^{n_i}$, for $i \leq m$, such that the structure $(\mathbb{U}, F_{n_1}, \dots, F_{n_m})$ is universal and ultrahomogeneous and it is unique (up to isometry preserving the relations) with this property.*

Proof. Consider the countable class \mathcal{K}_1 of finite rational metric structures defined as follows. We have $A \in \mathcal{K}_1$ if the following is satisfied:

- A is a finite rational metric space

- for each $i \leq m$ there is a 1-Lipschitz rational function $p_i : A^{n_i} \rightarrow \mathbb{Q}_0^+$ which interprets as a distance function from the desired closed set F_{n_i}

The class of embeddings consists of *all* isometric embeddings between elements of \mathcal{K} that preserve the functions p_i . We shall thus write just \mathcal{K}_1 instead of full $(\mathcal{K}_1, \mathcal{E}_1)$.

To check that \mathcal{K}_1 is a Fraïssé class we have to verify that it is countable and has the joint and amalgamation properties. Since all the functions take values in rationals, it is countable.

We check the amalgamation property. The joint embedding property is similar, only easier. Let $A, B, C \in \mathcal{K}_1$ be structures such that A is a common substructure of both B and C . The underlying metric space of the amalgam D will be the greatest metric amalgamation of B and C over A as defined in Definition 1.3. For each $i \leq m$ we have to extend p_{n_i} from $B^{n_i} \cup C^{n_i}$ onto D^{n_i} . We take the greatest 1-Lipschitz extension of p_{n_i} ; i.e. for any $\vec{d} \in D^{n_i}$ we set

$$p_{n_i}(\vec{d}) = \min\{p_{n_i}(\vec{a}) + d(\vec{d}, \vec{a}) : \vec{a} \in B^{n_i} \cup C^{n_i}\}.$$

That is a standard way of extension of real valued (resp. in this case, rational valued) Lipschitz functions, so we leave to the reader to check that it works.

It follows that \mathcal{K}_1 has a Fraïssé limit. The limit is a countable rational metric space M together with rational 1-Lipschitz functions $\mathbb{Q}P_{n_i} : M \rightarrow \mathbb{Q}_0^+$, for every $i \leq m$. However, the metric space M is actually the rational Urysohn space \mathbb{QU} . That immediately follows from Fact 1.2 that M has the metric rational one-point extension property. Denote also by $\mathbb{Q}F_{n_i}$, for each $i \leq m$, the set $\{x \in \mathbb{QU}^{n_i} : \mathbb{Q}P_{n_i}(x) = 0\}$. It follows from the Fraïssé theorem that the metric structure $(\mathbb{QU}, \mathbb{Q}P_{n_1}, \dots, \mathbb{Q}P_{n_m})$ has the following universality and *rational one-point extension* properties:

- for every $A \in \mathcal{K}_1$ there exists an embedding $\iota : A \hookrightarrow (\mathbb{QU}, \mathbb{Q}P_{n_1}, \dots)$, i.e. for each $i \leq m$ and for every $\vec{a} \in A^{n_i}$ we have $p_{n_i}(\vec{a}) = \mathbb{Q}P(\iota(\vec{a}))$
- for every $A \in \mathcal{K}_1$, every embedding $\iota : A \hookrightarrow (\mathbb{QU}, \mathbb{Q}P_{n_1}, \dots)$, and every one-point extension $A \subseteq B \in \mathcal{K}_1$ there exists an extension $\iota \subseteq \tilde{\iota} : B \hookrightarrow (\mathbb{QU}, \mathbb{Q}P_{n_1}, \dots)$

We now take the completion of this rational structure to obtain $(\mathbb{U}, P_{n_1}, \dots, P_{n_m})$, where \mathbb{U} is the Urysohn space, the completion of \mathbb{QU} , and for each $i \leq m$, P_{n_i} is the unique extension of $\mathbb{Q}P_{n_i}$ onto \mathbb{U} . Denote also by F_{n_i} the set $\overline{\mathbb{Q}F_{n_i}} = \{x \in \mathbb{U}^{n_i} : P_{n_i}(x) = 0\}$, i.e. the completion of $\mathbb{Q}F_{n_i}$ in \mathbb{U}^{n_i} .

Let $\overline{\mathcal{K}_1}$ denote the ‘real’ version of \mathcal{K}_1 , i.e. finite structures of the same type as those in \mathcal{K}_1 with the difference that we allow all functions there, including the metric, to take values in all reals, not only in rationals. In order to prove Theorem 2.1 we show that $(\mathbb{U}, P_{n_1}, \dots, P_{n_m})$ has the same universality and one-point extension properties with respect to $\overline{\mathcal{K}_1}$ as $(\mathbb{Q}\mathbb{U}, \mathbb{Q}P_{n_1}, \dots)$ does with respect to \mathcal{K}_1 . That is the content of the next lemma.

Lemma 2.2. *For every $A \in \overline{\mathcal{K}_1} \cup \{\emptyset\}$, every embedding $\iota : A \hookrightarrow (\mathbb{U}, P_{n_1}, \dots)$, and every one-point extension $A \subseteq B \in \overline{\mathcal{K}_1}$ there exists an extension $\iota \subseteq \tilde{\iota} : B \hookrightarrow (\mathbb{U}, P_{n_1}, \dots)$.*

By allowing A in the lemma to be empty, we get also the ‘real version’ of universality property, i.e. that for every $A \in \overline{\mathcal{K}_1}$ there is an embedding $\iota : A \hookrightarrow (\mathbb{U}, P_{n_1}, \dots)$. Once Lemma 2.2 is proved, we are done. Indeed, the universality, ultrahomogeneity and uniqueness of $(\mathbb{U}, P_{n_1}, \dots)$ follow from it by the similar arguments for proving universality, ultrahomogeneity and uniqueness of the Urysohn space using the one-point extension property. For a reader not familiar with such arguments, we provide a sketch of the proof of universality.

Let $(X, G_{n_1}, \dots, G_{n_m})$ be a Polish metric structure where X is a Polish metric space and for each $i \leq m$, G_{n_i} is a closed subset of X^{n_i} . Denote also by $Q_{n_i} : X^{n_i} \rightarrow \mathbb{R}_0^+$ the distance function from G_{n_i} . Let $D = \{d_n : n \in \mathbb{N}\} \subseteq X$ be some countable dense subset. Using Lemma 2.2 inductively, we build an increasing sequence of embeddings $\iota_1 \subseteq \iota_2 \subseteq \dots$, where for each j , $\iota_j : \{d_1, \dots, d_j\} \hookrightarrow (\mathbb{U}, P_{n_1}, \dots)$ is isometric and such that for each $i \leq m$ and every $\vec{d} \in \{d_1, \dots, d_j\}^{n_i}$ we have $Q_{n_i}(\vec{d}) = P_{n_i}(\iota_j(\vec{d}))$. Then we take $\iota = \bigcup_j \iota_j : D \hookrightarrow (\mathbb{U}, P_{n_1}, \dots)$. Since ι is an isometry, we can extend it to $\bar{\iota} : X \hookrightarrow (\mathbb{U}, P_{n_1}, \dots)$. Since for each $i \leq m$, Q_{n_i} is 1-Lipschitz, we also get that for every $\vec{x} \in X^{n_i}$ we have that $Q_{n_i}(\vec{x}) = P_{n_i}(\bar{\iota}(\vec{x}))$. In particular, for each $\vec{x} \in X^{n_i}$ we have that $\vec{x} \in G_{n_i}$ if and only if $\bar{\iota}(\vec{x}) \in F_{n_i}$, and we are done. Homogeneity and uniqueness are done similarly. We refer the reader again to Chapter 5 in [11] where these facts are proved for the plain Urysohn space.

Thus we are left to prove Lemma 2.2. We will do it in a series of three claims. We need one definition before.

Definition 2.3. We say that a class $K \subseteq \overline{\mathcal{K}_1}$ has an almost-one-point extension property if for every $A \in K$, every one-point extension $A \subseteq B = A \amalg \{b\} \in \overline{\mathcal{K}_1}$ and every $\varepsilon > 0$ there exists an extension $B' = A \amalg \{b'\} \in K$ such that for every $a \in A$ we have $|d(a, b) - d(a, b')| < \varepsilon$,

and for every $i \leq m$ and $\vec{a} \in B^{n_i}$ we have $|p_{n_i}(\vec{a}) - p_{n_i}(\vec{a}')| < \varepsilon \cdot n_i$, where \vec{a}' is obtained from \vec{a} by replacing each occurrence of b by b' .

Analogously, we say that a substructure $(U, P_{n_1} \upharpoonright U^{n_1}, \dots, P_{n_m} \upharpoonright U^{n_m})$ has an almost-one-point extension property if for every $A \in \overline{\mathcal{K}}_1$, every embedding $\iota : A \hookrightarrow (U, P_{n_1} \upharpoonright U^{n_1}, \dots)$, every $\varepsilon > 0$ and every one-point extension $A \subseteq B = A \coprod \{b\} \in \overline{\mathcal{K}}_1$ there exists an extension $B' = \iota[A] \coprod \{b'\} \subseteq (U, P_{n_1} \upharpoonright U^{n_1}, \dots)$ such that for every $a \in A$ we have $|d(a, b) - d(\iota(a), b')| < \varepsilon$, and for every $i \leq m$ and $\vec{a} \in B^{n_i}$ we have $|p_{n_i}(\vec{a}) - p_{n_i}(\vec{a}')| < \varepsilon \cdot n_i$, where \vec{a}' is obtained from \vec{a} by replacing each occurrence of b by b' and each occurrence of a_j by $\iota(a_j)$, for every $j \leq n$.

Claim 2.4. \mathcal{K}_1 has the almost-one-point extension property.

Claim 2.5. If a (not necessarily complete) substructure $(U, P_{n_1} \upharpoonright U^{n_1}, \dots, P_{n_m} \upharpoonright U^{n_m}) \subseteq (\mathbb{U}, P_{n_1}, \dots)$ has an almost-one-point extension property, then so does its completion.

Claim 2.6. If a complete substructure $(U, P_{n_1} \upharpoonright U^{n_1}, \dots, P_{n_m} \upharpoonright U^{n_m}) \subseteq (\mathbb{U}, P_{n_1}, \dots)$ has an almost-one-point extension property, then it has the one-point extension property.

Lemma 2.2 follows from these claims. Note at first, that the statement that \mathcal{K}_1 has an almost-one-point extension property is equivalent with the statement that $(\mathbb{Q}\mathbb{U}, \mathbb{Q}P_{n_1}, \dots)$ has an almost-one-point extension property. By Claim 2.4, $(\mathbb{Q}\mathbb{U}, \mathbb{Q}P_{n_1}, \dots)$ has an almost-one-point extension property. Then by Claim 2.5, its completion, $(\mathbb{U}, P_{n_1}, \dots)$ has also an almost-one-point extension property and using Claim 2.6 it must thus have the one-point extension property.

Let us now prove Claim 2.4. Let $A \in \mathcal{K}_1$ and $\varepsilon > 0$ and let $A \subseteq B \in \overline{\mathcal{K}}_1$ be some one-point extension. Let $f : A \rightarrow \mathbb{R}^+$ be the function $d(\cdot, b)$. Enumerate A as a_1, \dots, a_n so that we have $f(a_1) \geq f(a_2) \geq \dots \geq f(a_n)$. For each $j \leq n$ let q_j be an arbitrary rational number in the interval $(f(a_j) + (j-1) \cdot \varepsilon/n, f(a_j) + j \cdot \varepsilon/n]$. Let $B' = A \coprod \{b'\}$ be a one-point extension of A such that for every $j \leq n$ we have $d(a_j, b') = q_j$. The triangle inequalities are satisfied. Indeed, for every $j < k \leq n$ we have

$$|d(a_j, b') - d(a_k, b')| = |q_j - q_k| = q_j - q_k \leq f(a_j) - f(a_k) \leq d(a_j, a_k) \leq f(a_j) + f(a_k) \leq q_j + q_k = d(a_j, b') + d(a_k, b').$$

For each $i \leq m$ we also have to define p_{n_i} on $(B')^{n_i} \setminus A^{n_i}$. Note that for every $a \in A$ we have $d(a, b) < d(a, b')$, thus let $\delta = \min\{d(a, b') - d(a, b) : a \in A\}$. For every $\vec{a} \in B^{n_i} \setminus A^{n_i}$ let \vec{a}' be the corresponding

tuple from $(B')^{n_i} \setminus A^{n_i}$, i.e. where each occurrence of b is replaced by b' . For each $\vec{a} \in B^{n_i} \setminus A^{n_i}$ we set $p_{n_i}(\vec{a}')$ to be an arbitrary rational in the interval $[p_{n_i}(\vec{a}), p_{n_i}(\vec{a}) + \delta/2]$. Then for any $\vec{a}, \vec{b} \in B^{n_i} \setminus A^{n_i}$ we have

$$|p_{n_i}(\vec{a}') - p_{n_i}(\vec{b}')| \leq |p_{n_i}(\vec{a}) - p_{n_i}(\vec{b})| + \delta \leq d(\vec{a}, \vec{b}) + \delta \leq d(\vec{a}', \vec{b}')$$

verifying that p_{n_i} is 1-Lipschitz. It follows that $B' \in \mathcal{K}_1$ and it is as desired. That finishes the proof of Claim 2.4.

Proof of Claim 2.5 is the same as the proof of Lemma 5.1.15 in [11] and proof of Claim 2.6 is the same as the proof of Lemma 5.1.16 in [11], thus we refer the reader there. \square

Theorem 2.7. *There exists a universal and ultrahomogeneous 1-Lipschitz retraction $R : \mathbb{U} \rightarrow F_{\mathbb{U}} \subsetneq \mathbb{U}$, where $F_{\mathbb{U}}$ is again isometric to \mathbb{U} .*

Proof. Here we consider the following countable class \mathcal{K}_2 of finite rational metric structures. A finite structure A belongs to \mathcal{K}_2 if

- A is a rational metric space
- there is a rational 1-Lipschitz function $p : A \rightarrow \mathbb{Q}_0^+$ that interprets as a distance function from the desired universal retract
- there is a 1-Lipschitz retraction $r : A \rightarrow A_F$, where $A_F = \{a \in A : p(a) = 0\}$

We again consider all the embeddings that preserves metric (i.e. they are isometric) and functions p and r .

It is clear that \mathcal{K}_2 is countable. We again show just the amalgamation property. Suppose we have structures $A, B, C \in \mathcal{K}_2$, where we assume that A is a common substructure of both B and C . As in the proof of Theorem 2.1, we take the greatest metric amalgam D of B and C over A and show that it works. All we need to do is to check that p and r on D are still 1-Lipschitz which is analogous to the proof that functions p_{n_i} 's on amalgamans remain 1-Lipschitz from the proof of the previous theorem.

Thus we get some a Fraïssé limit $(\mathbb{Q}\mathbb{U}, \mathbb{Q}R, \mathbb{Q}P)$, where again $\mathbb{Q}\mathbb{U}$ is the limit as a metric space which is again the rational Urysohn space. $\mathbb{Q}R$ is the limit of the retractions and $\mathbb{Q}P$ is the limit of the distance function p . By $F_{\mathbb{Q}\mathbb{U}} \subseteq \mathbb{Q}\mathbb{U}$ we denote the set $\{a \in \mathbb{Q}\mathbb{U} : \mathbb{Q}P(a) = 0\} = \{a \in \mathbb{Q}\mathbb{U} : \mathbb{Q}R(a) = a\}$, i.e. the universal retract.

From the Fraïssé theorem we have the following universality and rational one-point extension property of $(\mathbb{Q}\mathbb{U}, \mathbb{Q}R, \mathbb{Q}P)$:

- for every $(A, r, p) \in \mathcal{K}_2$ we have an embedding $\iota : A \hookrightarrow (\mathbb{Q}\mathbb{U}, \mathbb{Q}R, \mathbb{Q}P)$, i.e. for every $a \in A$ we have $\mathbb{Q}R \circ \iota(a) = \iota \circ r(a)$ and $\mathbb{Q}P \circ \iota(a) = p(a)$,

- for every $A \in \mathcal{K}_2$, every embedding $\iota : A \hookrightarrow (\mathbb{Q}\mathbb{U}, \mathbb{Q}R, \mathbb{Q}P)$, and every one-point extension $A \subseteq B \in \mathcal{K}_2$ there exists an extension $\iota \subseteq \tilde{\iota} : B \hookrightarrow (\mathbb{Q}\mathbb{U}, \mathbb{Q}R, \mathbb{Q}P)$

We take the completion to obtain the Urysohn space together with the retraction R from \mathbb{U} onto $F_{\mathbb{U}} = \overline{F_{\mathbb{Q}\mathbb{U}}}$, which is the unique extension of $\mathbb{Q}R$, and with the distance function P , which is the unique extension of $\mathbb{Q}P$. The following lemma will finish the proof of the theorem.

Lemma 2.8. *(\mathbb{U}, R, P) has a one-point extension property, i.e. for every finite subset $A \subseteq \mathbb{U}$ and for every abstract one-point extension $B = A \coprod \{b\}$ there is a corresponding ‘concrete’ one-point extension in (\mathbb{U}, R, P) .*

Let us formulate the almost-one-point extension property in this situation. $\overline{\mathcal{K}_2}$ denotes the real finite substructures of the same type as those in \mathcal{K}_2 . That is an equivalent definition to $\overline{\mathcal{K}_1}$ in the proof of the previous theorem.

We say that a class $K \subseteq \overline{\mathcal{K}_2}$ has an almost-one-point extension property if for every $A \in K$, every extension $A \subseteq B = A \coprod \{b\} \in \overline{\mathcal{K}_2}$ and every $\varepsilon > 0$ there exists a (one or two point) extension $B' = A \coprod \{b', R(b')\} \in K$ such that for every $a \in A$ we have $|d(a, b) - d(a, b')| < \varepsilon$, $|p(b) - p(b')| < \varepsilon$, and if $R(b) \neq b$, i.e. $R(b) \in A$, then we have $d(R(b'), R(b)) < \varepsilon$. Analogous definition is used for substructures of $(\mathbb{U}, R, F_{\mathbb{U}})$ (as in Definition 2.3).

The following series of claims suffices to prove as in the proof of Theorem 2.1.

Claim 2.9. *\mathcal{K}_2 has the almost-one-point extension property.*

Claim 2.10. *If a (not necessarily complete) substructure $(U, R \upharpoonright U, P \upharpoonright U) \subseteq (\mathbb{U}, R, P)$ has an almost-one-point extension property, then so does its completion.*

Claim 2.11. *If a complete substructure $(U, R \upharpoonright U, P \upharpoonright U) \subseteq (\mathbb{U}, R, P)$ has an almost-one-point extension property, then it has the one-point extension property.*

Claim 2.9, resp. Claim 2.11, is proved as Claim 2.4, resp. Claim 2.6. We only prove Claim 2.10. Let $(U, R \upharpoonright U, P \upharpoonright U) \subseteq (\mathbb{U}, R, P)$ be some not-complete substructure having an almost-one-point extension property and let $(\bar{U}, R \upharpoonright \bar{U}, P \upharpoonright \bar{U})$ be its completion. We note here that since U is a substructure, it is closed under R , i.e. for every $u \in U$ we have $R(u) \in U$. We show that $(U, R \upharpoonright U, P \upharpoonright U)$ has the almost-one-point extension property. Let A be some finite substructure of $(\bar{U}, R \upharpoonright \bar{U}, P \upharpoonright \bar{U})$ and let $A \subseteq B = A \coprod \{b\} \in \overline{\mathcal{K}_2}$ be its one-point

extension. Let $\varepsilon > 0$ be given. Since U is dense in \bar{U} we can find for each $a \in A$ an element $u(a) \in U$ such that $d(u(a), a) < \varepsilon/2$ and so that for $a_1 \neq a_2$, $u(a_1) \neq u(a_2)$. We set $A' = \{u(a) : a \in A\} \cup \{R(u(a)) : a \in A\}$, i.e. closing $\{u(a) : a \in A\}$ under R . Note that it is possible that $|A'| > |A|$ since there might be $a \neq b \in A$ such that $R(a) = R(b)$, however $R(u(a)) \neq R(u(b))$. Nevertheless, since R is 1-Lipschitz we have that $d(R(a), R(u(a))) < \varepsilon/2$ for every $a \in A$. Now take the (greatest) metric amalgamation of $A' \cup A$ with B over A . That gives some metric on $A' \cup \{b\}$. If $R(b) = b$ in B then we define $B' = A' \cup \{b\}$ to be the extension of A' where also $R(b) = b$; that gives some structure from $\overline{\mathcal{K}_2}$. Otherwise, $R(b) = a$ for some $a \in A \subseteq B$ in B . Then we define $B' = A' \cup \{b\}$ to be the extension of A' where $R(b) = u(a)$; that also gives some structure from $\overline{\mathcal{K}_2}$.

We may thus find some almost-one-point extension for $\varepsilon/2$ in U . It is straightforward to check that it will be an almost-one-point extension for A as well. \square

Theorem 2.12. *Let K be an arbitrary compact metric space. Then there exists a closed subset $C \subseteq \mathbb{U} \times K$ that is universal and ultrahomogeneous.*

Proof. Let $D_K = \{q_n : n \in \mathbb{N}\}$ be an enumeration of some countable dense subset of K . For some metric space M we want to consider a function $f : M \times \mathbb{N} \rightarrow \mathbb{R}_0^+$, where $f(x, n)$, for any $x \in M$ and $n \in \mathbb{N}$, is to be interpreted as the distance (again in the sum metric) of (x, q_n) from some closed subset of $M \times K$. However, even if we restrict to finite rational metric spaces and demand that f takes only rational values, we still get uncountably many possible distance functions. The remedy is to consider distance functions that are ‘controlled’ by finite sets. Let A be a finite rational metric space. A rational distance function $f : A \times \mathbb{N} \rightarrow \mathbb{Q}_0^+$, i.e. 1-Lipschitz on $A \times D_K$, is called *finitely-controlled* if there exists a finite subset $N \subseteq \mathbb{N}$ such that for any $a \in A$ and $n \in \mathbb{N}$ we have $f(a, n) = \max\{0, \max\{f(a, m) - d_K(q_m, q_n) : m \in N\}\}$. Clearly, for any rational metric space A there are only countably many finitely-controlled distance functions.

We shall consider a countable class of finite rational metric structures \mathcal{K}_3 such that A belongs to \mathcal{K}_3 if

- A is a finite rational metric space
- A is equipped with a rational 1-Lipschitz finitely-controlled distance function $f : A \times \mathbb{N} \rightarrow \mathbb{Q}_0^+$

\mathcal{K}_3 is countable. Joint embedding for two structures $A, B \in \mathcal{K}_3$ can be achieved by putting A and B far apart from each other. Amalgamation

of two structures $B, C \in \mathcal{K}_3$ over a third structure $A \in \mathcal{K}_3$ can be again achieved by taking the greatest metric amalgam of B and C over A . f is correctly defined on the amalgam since it is 1-Lipschitz; that follows from the same argument as in Theorems 2.1 and 2.7.

Thus we get a Fraïssé limit $(\mathbb{Q}\mathbb{U}, \mathbb{Q}F)$, where $\mathbb{Q}F : \mathbb{Q}\mathbb{U} \times \mathbb{N} \rightarrow \mathbb{Q}_0^+$ is the limit of the distance function. By $\mathbb{Q}C \subseteq \mathbb{Q}\mathbb{U} \times K$ we shall denote the set $\{(u, q_n) : \mathbb{Q}F(u, n) = 0\}$. We have the following universality and rational one-point extension property of $(\mathbb{Q}\mathbb{U}, \mathbb{Q}F)$:

- for every $A \in \mathcal{K}_3$ there exists an embedding $\iota : A \hookrightarrow (\mathbb{Q}\mathbb{U}, \mathbb{Q}F)$, i.e. for every $a \in A$ and $n \in \mathbb{N}$ we have $f(a, n) = \mathbb{Q}F(\iota(a), n)$
- for every $A \in \mathcal{K}_3$, every embedding $\iota : A \hookrightarrow (\mathbb{Q}\mathbb{U}, \mathbb{Q}F)$, and every one-point extension $A \subseteq B \in \mathcal{K}_3$ there exists an extension $\iota \subseteq \tilde{\iota} : B \hookrightarrow (\mathbb{Q}\mathbb{U}, \mathbb{Q}F)$

We take the completion of $(\mathbb{Q}\mathbb{U}, \mathbb{Q}F)$ to get (\mathbb{U}, F) , where is the unique extension of $\mathbb{Q}F$ onto $\mathbb{U} \times \mathbb{N}$. Also, by $C \subseteq \mathbb{U} \times K$ we denote the closure of $\mathbb{Q}C$ in $\mathbb{U} \times K$. By $\overline{\mathcal{K}_3}$ we shall denote the class of finite metric spaces M with 1-Lipschitz distance function $f : M \times \mathbb{N} \rightarrow \mathbb{R}_0^+$ that not only can take values in reals, it does not have to be finitely-controlled.

We say that a subclass $K \subseteq \overline{\mathcal{K}_3}$ has an almost-one-point extension property if for every $A \in K$, every $\varepsilon > 0$ and every one-point extension $A \subseteq B = A \amalg \{b\} \in \overline{\mathcal{K}_3}$ there exists a one-point extension $B' = A \amalg \{b'\} \in K$ such that for every $a \in A$ and every $n \in \mathbb{N}$ we have $|d(a, b) - d(a, b')| < \varepsilon$ and $|f(b, n) - f(b', n)| < \varepsilon$. Analogously, we define an almost-one-point extension property for a substructure $(U, F) \subseteq (\mathbb{U}, F)$.

As before we need to prove:

Lemma 2.13. *(\mathbb{U}, F) has the one-point extension property.*

That will be again achieved by proving the following three claims.

Claim 2.14. *\mathcal{K}_3 has an almost-one-point extension property.*

Claim 2.15. *If a (not necessarily complete) substructure $(U, F \upharpoonright U \times \mathbb{N}) \subseteq (\mathbb{U}, F)$ has an almost-one-point extension property, then so does its completion.*

Claim 2.16. *If a complete substructure $(U, F \upharpoonright U \times \mathbb{N}) \subseteq (\mathbb{U}, F)$ has an almost-one-point extension property, then it has the one-point extension property.*

Claims 2.15 and 2.16 are proved similarly as the analogous Claims 2.5 and 2.6. We only prove Claim 2.14.

Fix $A \in \mathcal{K}_3$, $\varepsilon > 0$ and a one-point extension $B = A \amalg \{b\} \in \overline{\mathcal{K}_3}$. We define the rational metric on B' precisely as we did in the proof

of the analogous Claim 2.4. It remains to define the finitely controlled rational distance function f on b' . Note again that we have that for every $a \in A$, $d(a, b) < d(a, b')$ and let $\delta = \min\{\min\{d(a, b') - d(a, b) : a \in A\}, \varepsilon/3\}$. Since K is compact there exists a finite set $N \subseteq \mathbb{N}$ such that $\{q_n : n \in N\}$ is a $\varepsilon/3$ -net in K . We define a finitely-controlled f controlled by values on N (we may without loss of generality assume that f on $A \times \mathbb{N}$ was controlled by values on N). For each $n \in N$ we set $f(b', n)$ to be an arbitrary rational number from the interval $[f(b, n), f(b, n) + \delta]$. For other $m \in \mathbb{N}$ it extends uniquely by the formula $f(b', m) = \max\{0, \max\{f(b', n) - d_K(q_n, q_m) : n \in N\}\}$. We check that f is still 1-Lipschitz, i.e. for any $a \in A$ and n, m we have $|f(a, n) - f(b', m)| \leq d(a, b') + d_K(q_n, q_m)$. Since f is controlled by values on N it suffices to consider n, m from N . We have

$$\begin{aligned} |f(a, n) - f(b', m)| &\leq |f(a, n) - f(b, m)| + \delta \leq \\ &d(a, b) + d_K(q_n, q_m) + \delta \leq d(a, b') + d_K(q_n, q_m). \end{aligned}$$

Finally, we need to check that for any $n \in \mathbb{N}$ we have $|f(b, n) - f(b', n)| \leq \varepsilon$. It is clear for $n \in N$, so let $n \in \mathbb{N}$ be arbitrary. However, since $\{q_n : n \in N\}$ forms an $\varepsilon/3$ -net in K there exists $m \in N$ such that $d_K(q_n, q_m) < \varepsilon/3$. Then we have

$$\begin{aligned} |f(b, n) - f(b', n)| &\leq |f(b, n) - f(b, m)| + |f(b, m) - f(b', m)| + \\ &|f(b', m) - f(b', n)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \end{aligned}$$

since f is 1-Lipschitz. This finishes the proof. \square

Theorem 2.17. *Let Z be an arbitrary Polish metric space and $L > 0$ an arbitrary real constant. Then there exists an L -Lipschitz function $F : \mathbb{U} \rightarrow Z$ that is universal and ultrahomogeneous.*

Proof. Let $D_Z = \{z_n : n \in \mathbb{N}\} \subseteq Z$ be some countable dense subset of Z . We consider the following countable class \mathcal{K}_4 of finite structures. We have that $A \in \mathcal{K}_4$ if

- A is a finite rational metric space
- there is an L -Lipschitz function $p : A \rightarrow D_Z$

\mathcal{K}_4 is clearly countable and has the hereditary property. Joint embedding for two structures $A, B \in \mathcal{K}_4$ can be again obtained by putting A and B ‘far apart’ from each other; i.e. if $m = \max\{L \cdot d_Z(p(a), p(b)) : a, b \in A \amalg B\}$ and M is some rational greater than m and $\text{diam}(A)$ and $\text{diam}(B)$ then we can take the disjoint union $A \amalg B$ and define the distance between any $a \in A$ and $b \in B$ to be $2M$. This works. We show the amalgamation property. Suppose we have structures $A, B, C \in \mathcal{K}_4$, where we assume that A is a common substructure of both B and C . As usual, we take the greatest metric amalgam of B and C over

A and show that it works. We need to show that for $b \in B$ and $c \in C$ we have $d_Z(p(b), p(c)) \leq L \cdot d(b, c)$. Let $a \in A$ be such that $d(b, c) = d(b, a) + d(a, c)$. Then we have

$$\begin{aligned} d_Z(p(b), p(c)) &\leq d_Z(p(b), p(a)) + d_Z(p(a), p(c)) \leq \\ &L \cdot (d(b, a) + d(a, c)) = L \cdot d(b, c) \end{aligned}$$

and we are done.

We now consider the Fraïssé limit. As before, it is easy to check that it is the rational Urysohn space \mathbb{QU} together with an L -Lipschitz function $\mathbb{QP} : \mathbb{QU} \rightarrow D_Z$. The following universal property and rational one-point extension property characterize $(\mathbb{QU}, \mathbb{QP})$:

- for every $A \in \mathcal{K}_4$ there exists an embedding $\iota : A \hookrightarrow (\mathbb{QU}, \mathbb{QP})$, i.e. ι is isometric and for every $a \in A$ we have $p(a) = \mathbb{QP} \circ \iota(a)$,
- for every $A \in \mathcal{K}_4$, every embedding $\iota : A \hookrightarrow (\mathbb{QU}, \mathbb{QP})$, and every one-point extension $A \subseteq B \in \mathcal{K}_4$ there exists an extension $\iota \subseteq \tilde{\iota} : B \hookrightarrow (\mathbb{QU}, \mathbb{QP})$

We take the completion (\mathbb{U}, P) , where $P : \mathbb{U} \rightarrow Z$ is the unique extension of \mathbb{QP} , the dense subset of \mathbb{U} , on the whole domain \mathbb{U} . To finish the proof, we must show the following.

Lemma 2.18. *(\mathbb{U}, P) has a one-point extension property, i.e. for every finite subset $A \subseteq \mathbb{U}$ and for every abstract one-point extension $(B = A \coprod \{b\}, \bar{p} : B \rightarrow Z)$, where $\bar{p} = P \upharpoonright A \cup \{(b, z) \text{ for } z \in Z\}$, there is a corresponding ‘concrete’ one-point extension in (\mathbb{U}, P) .*

By $\overline{\mathcal{K}_4}$ we mean the class of all finite metric spaces equipped with an L -Lipschitz function p with values in Z , not necessarily in the countable dense set D_Z .

We say that a class $K \subseteq \overline{\mathcal{K}_4}$ has an almost-one-point extension property if for every $A \in K$, every extension $A \subseteq B = A \coprod \{b\} \in \overline{\mathcal{K}_4}$ and every $\varepsilon > 0$ there exists a one-point extension $B' = A \coprod \{b'\} \in K$ such that for every $a \in A$ we have $|d(a, b) - d(a, b')| < \varepsilon$ and $d_Z(p(b), p(b')) < L \cdot \varepsilon$. The almost-one-point extension property for a substructure $(U, P \upharpoonright U) \subseteq (\mathbb{U}, P)$ is defined analogously as in the proofs of Theorems 2.1, 2.7 and 2.12. Again note that an almost-one-point extension property for \mathcal{K}_4 is equivalent with an almost-one-point extension property for $(\mathbb{QU}, \mathbb{QP})$.

As before, we prove Lemma 2.18 through the following series of claims.

Claim 2.19. *\mathcal{K}_4 has the almost-one-point extension property.*

Claim 2.20. *If a (not necessarily complete) substructure $(U, P \upharpoonright U) \subseteq (\mathbb{U}, P)$ has an almost-one-point extension property, then so does its completion.*

Claim 2.21. *If a complete substructure $(U, P \upharpoonright U) \subseteq (\mathbb{U}, P)$ has an almost-one-point extension property, then it has the one-point extension property.*

We shall only prove Claim 2.19, the proofs of the other two claims are routine and modifications of the analogous ones from the proofs of Theorems 2.1, 2.7 and 2.12.

Let $A \in \mathcal{K}_4$ and $A \subseteq B \in \overline{\mathcal{K}_4}$ with some $\varepsilon > 0$ be given, where $B = A \coprod \{b\}$. We define $A \subseteq B' \in \mathcal{K}_4$ as follows. $B' = A \coprod \{b'\}$ and for every $a \in A$ we define $d(a, b')$ as in Claim 2.4. In particular, we again have that

$$(2.1) \quad d(a, b') > d(a, b)$$

for every $a \in A$. If $p(b) \in D_Z$, then we set $p(b') = p(b)$ and because of (2.1) we have $d_Z(p(a) - p(b')) = d_Z(p(a), p(b)) \leq L \cdot d(a, b) \leq d(a, b')$ for every $a \in A$ and we are done. If $p(b) \notin D_Z$, in particular $p(b)$ is not an isolated point of Z , we choose some $z_0 \in D_Z$ such that $d_Z(z_0, p(b)) < L \cdot \delta$, where $\delta = \min\{d(a, b') - d(a, b) : a \in A\} > 0$. Then we put $p(b') = z_0$ and we claim that this works. Indeed, for any $a \in A$ we have

$$d_Z(p(a), z_0) \leq d_Z(p(a), p(b)) + L \cdot \delta \leq L \cdot (d(a, b) + \delta) \leq L \cdot d(a, b').$$

This finishes the proof of Theorem 2.17. \square

We have an interesting corollary of Theorem 2.17. This theorem ‘lifts’ from the category of complete metric spaces to the category of Banach spaces via the functor assigning to a metric space its Lipschitz-free Banach space and to a Lipschitz map its linear extension. We refer the reader to the book [13] for information about Lipschitz-free Banach spaces and to the paper [4] of Godefroy and Kalton. Here we just recall that for every metric space X (with a distinguished point representing 0) there exists a Banach space $F(X)$ in which there is an isometric copy of X such that $\text{span}\{X\}$ is dense in $F(X)$ and that is uniquely characterized by the property that for every Banach space Y and Lipschitz map $f : X \rightarrow Y$ sending 0 to 0 there exists a unique bounded linear operator with the same Lipschitz constant $\hat{f} : F(X) \rightarrow Y$ that extends f .

In case X is a Banach space and $\iota : X \rightarrow F(X)$ is the canonical isometric embedding, then by $\beta : F(X) \rightarrow X$ we denote ι^{-1} , i.e. the unique linear operator from $F(X)$ to X such that we have $\beta \circ \iota = \text{id}$.

The following theorem was proved in [4].

Theorem 2.22 (Godefroy, Kalton [4]). *For any separable Banach space X there exists a linear isometry $\iota_{GK} : X \rightarrow F(X)$ such that $\beta \circ \iota_{GK} = \text{id}_X$.*

It follows that the so-called Holmes space \mathbb{H} , the Lipschitz-free space $F(\mathbb{U})$ over the Urysohn space, is a universal separable Banach space. We refer the reader to the paper of Holmes [7] and to Chapter 5 of [11] for more information about this Banach space. We remark that it was proved by Fonf and Wojtaszczyk in [2] that the Holmes universal space is not linearly isometric to other known universal Banach spaces such as the Gurarij space or $C([0, 1])$. It is also not isomorphic to the Pełczyński universal space which also follows from the results from [2].

We note that universal and homogeneous linear operator on the Gurarij space was constructed in [3].

Theorem 2.23. *Let Z be an arbitrary separable Banach space and $L > 0$ an arbitrary real constant. Then there exists a universal linear operator $\Phi : \mathbb{H} \rightarrow Z$ of norm L .*

Proof. Let $F : \mathbb{U} \rightarrow Z$ be the universal L -Lipschitz map from the Urysohn space to the Banach space Z . Denote by Φ the unique linear extension of F from \mathbb{U} to \mathbb{H} , where we chose 0 in \mathbb{U} so that $F(0) = 0$. We claim that Φ is as desired.

Indeed, let X be an arbitrary separable Banach space equipped with a linear operator $\psi : X \rightarrow Z$ such that $\|\psi\| \leq L$. Using Theorem 2.17 we obtain a (non-linear) isometric embedding $\iota : X \hookrightarrow \mathbb{U}$ such that for every $x \in X$ we have

$$(2.2) \quad F \circ \iota(x) = \psi(x)$$

We shall again denote by X' the image of X in \mathbb{U} and by $\beta : F(X') \subseteq \mathbb{H} \rightarrow X$ the canonical surjective linear operator from $F(X')$ onto X so that we have $\beta \circ \iota = \text{id}_X$. By Theorem 2.22 there exists a linear isometry $\iota_{GK} : X \rightarrow F(X') \subseteq \mathbb{H}$ such that $\beta \circ \iota_{GK} = \text{id}_X$. We claim that for every $x \in X$ we have $\Phi \circ \iota_{GK} = \psi$. Once we prove it we are done.

Consider the linear operator $\Phi' = \psi \circ \beta$. We have

$$\Phi' \circ \iota_{GK} = \psi \circ \beta \circ \iota_{GK} = \psi \circ \text{id}_X = \psi$$

thus it suffices to prove that $\Phi \upharpoonright F(X') = \Phi'$. However, $\Phi \upharpoonright F(X')$ is uniquely determined by the property that for every $x \in X'$ we have $\Phi(x) = F(x)$. But if we take any $x \in X'$ then

$$\Phi'(x) = \psi \circ \beta(x) = \psi \circ \iota^{-1}(x) = F(x)$$

where the last equality follows from (2.2) and we are done. \square

3. GURARIJ SPACE

Our aim in this section is to prove similar universality and homogeneity results that we did for the Urysohn space for the Gurarij space. We recall the definition of the Gurarij space.

Definition 3.1. Recall that a separable Banach space \mathbb{G} is Gurarij if it satisfies the following property: for every $\varepsilon > 0$, every finite dimensional Banach spaces $E \subseteq F$ and every linear isometry $\phi : E \hookrightarrow \mathbb{G}$ there exists an extension $\bar{\phi} \supseteq \phi : F \hookrightarrow \mathbb{G}$ such that $\bar{\phi}$ is an ‘ ε -isometry’, i.e. for every $x \in F$ we have $(1 - \varepsilon) \cdot \|x\| \leq \|\bar{\phi}(x)\| \leq (1 + \varepsilon) \cdot \|x\|$.

Before formulating the theorems, let us define a necessary notion. Let (S_E, E) and (S_F, F) be pairs of 1-Lipschitz seminorms together with Banach spaces. Let $\varepsilon > 0$. Then we say that $\phi : (S_E, E) \rightarrow (S_F, F)$ is an ε -*morphism* if ϕ is a linear ε -isometry between E and F in their norms and also an ε -isometry between the quotients of E and F by their respective seminorms, i.e. for every $x \in E$ we have $|S_E(x) - S_F(x)| < \varepsilon \cdot S_E(x)$.

Since we shall speak always about seminorms that are 1-Lipschitz, we may sometimes omit the adjective ‘1-Lipschitz’ when talking about them.

The following simple fact that we state without a proof shows why we consider 1-Lipschitz seminorms.

Fact 3.2. *Let X be a Banach space. There is a (not always one-to-one) correspondence between closed subspaces $Y \subseteq X$ and 1-Lipschitz seminorms $S : X \rightarrow \mathbb{R}_0^+$.*

Namely, the function $\text{dist}(\cdot, Y) : X \rightarrow \mathbb{R}_0^+$ is a 1-Lipschitz seminorm and the set $\{x \in X : S(x) = 0\}$ is a closed subspace.

We shall use the following notation: whenever X is a Banach space and $Y \subseteq X$ is a closed subspace, then by $S_X^Y : X \rightarrow \mathbb{R}_0^+$ we denote the seminorm $\text{dist}(\cdot, Y)$. Conversely, if $S : X \rightarrow \mathbb{R}_0^+$ is a 1-Lipschitz seminorm, then by $Y_S \subseteq X$ we denote the closed subspace $\{x \in X : S(x) = 0\}$.

The following theorems are the main results. The first one is analogous to Theorem 2.1, we only replace 1-Lipschitz function by 1-Lipschitz seminorms. The second one is analogous to Theorem 2.7, we only replace 1-Lipschitz retraction by 1-Lipschitz projection.

Theorem 3.3. *There exists a 1-Lipschitz seminorm $\mathbb{S} : \mathbb{G} \rightarrow \mathbb{R}_0^+$ on the Gurarij space such that the pair (\mathbb{S}, \mathbb{G}) has the following homogeneity property: for every finite dimensional Banach space E with a seminorm $S_E : E \rightarrow \mathbb{R}_0^+$ and for every finite dimensional extension F equipped*

with a seminorm $S_F : F \rightarrow \mathbb{R}_0^+$ that extends S_E , for every $\varepsilon > 0$ and for every 0-morphism $\phi : (S_E, E) \hookrightarrow (\mathbb{S}, \mathbb{G})$ there exists an extension $\bar{\phi} \supseteq \phi : (S_F, F) \hookrightarrow (\mathbb{S}, \mathbb{G})$ that is an ε -morphism.

Consequently, for every separable Banach space X with a closed subspace Y there exists a linear isometry $\phi : X \hookrightarrow \mathbb{G}$ such that for every $x \in X$ we have $\text{dist}(x, Y) = \text{dist}(\phi(x), \mathbb{H})$, where $\mathbb{H} = \{x \in \mathbb{G} : \mathbb{S}(x) = 0\}$ may be understood as a universal closed subspace of the Gurarij space.

If X is a Banach space, $P : X \rightarrow Y$ is a norm one projection onto its closed subspace and $S : X \rightarrow \mathbb{R}_0^+$ is a seminorm, then we say that P and S are *compatible* if $S \leq S_X^Y$ (i.e. for every $x \in X$, $S(x) \leq S_X^Y(x)$) and for every $x \in X$, $S(x) = S_X^Y(x)$.

Theorem 3.4. *There exists a norm one projection $P : \mathbb{G} \rightarrow \mathbb{H}$ onto a 1-complement subspace of the Gurarij space with the following homogeneity property: for every finite dimensional Banach space E together with a norm one projection $P_E : E \rightarrow E_0$ onto its 1-complemented subspace E_0 and a compatible seminorm S_X , and for every finite dimensional extension F equipped with a norm one projection $P_F \supseteq P_E : F \rightarrow F_0$ onto its 1-complemented subspace F_0 and a compatible seminorm $S_F \supseteq S_E$, for every $\varepsilon > 0$ and for every 0-morphism $\phi : (S_E, E) \hookrightarrow (S_{\mathbb{G}}^{\mathbb{H}}, \mathbb{G})$ such that for every $x \in E$ we have $\phi \circ P_E(x) = P \circ \phi(x)$ there exists an extension $\bar{\phi} \supseteq \phi : (S_F, F) \hookrightarrow (S_{\mathbb{G}}^{\mathbb{H}}, \mathbb{G})$ such that $\bar{\phi}$ is an ε -morphism and again for every $x \in F$ we have $\bar{\phi} \circ P_F(x) = P \circ \bar{\phi}(x)$.*

Consequently, for every separable Banach space X with a norm one projection $p : X \rightarrow X_0$ there exists a linear isometry $\phi : X \hookrightarrow \mathbb{G}$ such that for every $x \in X$ we have $\text{dist}(x, X_0) = \text{dist}(\phi(x), \mathbb{H})$ and $P \circ \phi(x) = \phi \circ p(x)$.

Before we prove these two theorems we present a new construction of the Gurarij space that is ‘Fraïssé like’ in the classical sense. Then we will be able to prove Theorems 3.3, resp. 3.4 using minor modifications of proofs of Theorems 2.1, resp. 2.7.

Definition 3.5. Let X be a vector space and let $A \subseteq X$ be a subset such that $\text{span}\{A\} = X$. A partial A -norm $\|\cdot\|_A$ is non-negative real function which behaves like a norm except that it is defined only on A ; i.e. for every $x, y \in A$ and $\alpha \in \mathbb{R}$ we have

- $\|x\|_A = 0$ iff $x = 0$,
- $\|\alpha \cdot x\|_A = |\alpha| \cdot \|x\|_A$ if $\alpha \cdot x \in A$,
- $\|x + y\|_A \leq \|x\|_A + \|y\|_A$ if $x + y \in A$.

Fact 3.6. *Let X be a vector space, $A \subseteq X$ a subset such that $\text{span}\{A\} = X$ and let $\|\cdot\|_A : A \rightarrow \mathbb{R}$ be a partial A -norm. Then for any $x \in X$*

the formula

$$\|x\|_X^A = \inf\{|\alpha_1| \cdot \|a_1\|_A + \dots + |\alpha_n| \cdot \|a_n\|_A : \\ a_1, \dots, a_n \in A, \alpha_1 \cdot a_1 + \dots + \alpha_n \cdot a_n = x\}$$

defines a maximal seminorm $\|\cdot\|_X^A$ on X that extends $\|\cdot\|_A$.

If X is finite dimensional and A is finite, then $\|\cdot\|_X^A$ is actually a norm and the infimum in the formula may be replaced by minimum.

Proof. It is easy to check that it is a seminorm that extends $\|\cdot\|_A$. Since every seminorm $\|\cdot\|$ extending $\|\cdot\|_A$ must satisfy the inequality $\|x\| \leq |\alpha_1| \cdot \|a_1\|_A + \dots + |\alpha_n| \cdot \|a_n\|_A$ for every $a_1, \dots, a_n \in A$ such that $\alpha_1 \cdot a_1 + \dots + \alpha_n \cdot a_n = x$, we have that $\|\cdot\|_X^A$ is maximal.

Now suppose that X is finite dimensional and that $A = \{a_1, \dots, a_n\}$. Fix some $x \in X$ and take some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that $\alpha_1 \cdot a_1 + \dots + \alpha_n \cdot a_n = x$. Let $\delta = |\alpha_1| \cdot \|a_1\|_A + \dots + |\alpha_n| \cdot \|a_n\|_A$, thus $\|x\|_X^A \leq \delta$. The following is a compact subset of \mathbb{R}^n : $K_x = \{(\beta_1, \dots, \beta_n) : \beta_1 \cdot a_1 + \dots + \beta_n \cdot a_n = x, |\beta_1| \cdot \|a_1\|_A + \dots + |\beta_n| \cdot \|a_n\|_A \leq \delta\}$. Moreover, the map $(\beta_1, \dots, \beta_n) \rightarrow |\beta_1| \cdot \|a_1\|_A + \dots + |\beta_n| \cdot \|a_n\|_A$ is continuous, thus attains the minimum at some tuple $(\beta_1, \dots, \beta_n) \in K_x$. It also follows that $\|\cdot\|_X^A$ is a norm. \square

Definition 3.7. Let \mathcal{K} be the following class of Banach spaces. We have that $X \in \mathcal{K}$ if:

- X is a finite dimensional vector space with a specified basis (x_1, \dots, x_n)
- the norm $\|\cdot\|$ on X is of the form $\|\cdot\|_X^A$, i.e. determined by a partial norm $\|\cdot\|_A$. Moreover, we demand that A is a finite subset of X containing the basis such that each element of A is a linear combination of elements of the basis using only rational scalars, and $\|\cdot\|_A : A \rightarrow \mathbb{Q}$ is a partial norm taking values only in rationals

The class of embeddings \mathcal{E} consists only of those linear isometric embeddings between elements of \mathcal{K} that send elements of basis to elements of basis; i.e. if $X, Y \in \mathcal{K}$, where the basis of X is (x_1, \dots, x_n) and the basis of Y is (y_1, \dots, y_m) , then an allowed linear isometric embedding from X into Y is determined by an injection $\iota : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$. We shall such linear embeddings *proper*.

Fact 3.8. \mathcal{K} is a Fraïssé class.

Proof. It immediately follows that \mathcal{K} is countable.

Let us check the amalgamation property. Let $X_0, X_1, X_2 \in \mathcal{K}$, where X_0 is a common subspace of X_1 and X_2 . Moreover, we have that (x_1, \dots, x_n) is a basis of X_0 , $(x_1, \dots, x_n, y_1, \dots, y_m)$ is a basis of X_1

and $(x_1, \dots, x_m, z_1, \dots, z_k)$ is a basis of X_2 . The amalgam space X_3 is algebraically nothing else but the amalgamated direct sum $X_1 \oplus_{X_0} X_2$ of X_1 and X_2 over X_0 . The norm is defined again in a standard way, i.e the amalgamation norm (analogous to the greatest metric amalgamation) $\|x - y\| = \inf\{\|x - z\|_{X_1} + \|z - y\|_{X_2} : z \in X_0\}$ for $x \in X_1, y \in X_2$. To check that $X_3 \in \mathcal{K}$ first observe that it is a vector space with basis $(x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_k)$. If $\|\cdot\|_{X_1}$ was given by some partial norm $\|\cdot\|_{A_1}$ for finite $A_1 \subseteq X_1$, and $\|\cdot\|_{X_2}$ was given by some partial norm $\|\cdot\|_{A_2}$ for finite $A_2 \subseteq X_2$, then considering A_1, A_2 as subsets of X_3 we can form a partial norm $\|\cdot\|_{A_3}$ defined on $A_3 = A_1 \cup A_2$ so that for $a \in A_3$, $\|a\|_{A_3}$ is equal to $\|a\|_{A_1}$ if $a \in A_1$ and equal to $\|a\|_{A_2}$ if $a \in A_2$. It is straightforward to check that this defines a partial norm on A_3 and that the extension $\|\cdot\|_{X_3}^{A_3}$ is the amalgam norm on X_3 . For a reader unfamiliar with amalgam metrics, resp. norms, we refer to our paper [1] where a similar fact was verified for norms (resp. invariant metrics) on abelian groups.

The joint embedding property is similar, only easier. \square

Thus there exists the Fraïssé limit G , a direct limit of some countable sequence $X_1 \rightarrow X_2 \rightarrow \dots$ from \mathcal{K} . The following extension property follows from the Fraïssé theorem.

Fact 3.9. *Let $Y \in \mathcal{K}$ and let there be a proper linear embedding $\phi : X_n \hookrightarrow Y$ for some n . Then there exists $m > n$ and a proper linear embedding $\psi : Y \hookrightarrow X_m$ such that $\psi \circ \phi = \subseteq_{n \rightarrow m}$, where $\subseteq_{n \rightarrow m}$ is the inclusion proper embedding from X_n into X_m .*

It is a separable normed space and we take the completion which we shall denote by \mathbb{G} .

Theorem 3.10. \mathbb{G} is the Gurarij space.

Proof. We need to check the condition from Definition 3.1. It is sufficient to prove the following:

Claim 3.11. *For every $\varepsilon > \varepsilon' > 0$, every finite dimensional Banach spaces $E \subseteq F$, where*

- *E is of co-dimension 1 in F , $F = \text{span}\{E, v\}$ and $v \in F \setminus E$ such that $\|v\| = 1$*
- *$\varepsilon' < \min\{\varepsilon/2, \frac{\varepsilon\delta}{10}\}$, where $\delta = \text{dist}(v, E) = \inf\{\|v - x\| : x \in E\}$*

we have that any ε' -isometry $\phi : E \hookrightarrow \mathbb{G}$ can be extended to $\bar{\phi} \supseteq \phi : F \hookrightarrow \mathbb{G}$ such that $\bar{\phi}$ is an ε -isometry.

Indeed, suppose that we have proved Claim 3.11 and we are given subspaces $E \subseteq F$, $\varepsilon > 0$ and an isometry $\phi : E \hookrightarrow \mathbb{G}$. Suppose

the co-dimension of E in F is n and E has basis $\{e_1, \dots, e_m\}$, which can be extended to a basis $\{e_1, \dots, e_m, f_1, \dots, f_n\}$ of F . Then the extension of ϕ to $\bar{\phi}$ is done using Claim 3.11 n -times through spaces $E = E_0 \leq \dots \leq E_i = \text{span}\{E, f_1, \dots, f_i\} \leq \dots E_n = F$ so that the extension $\phi_i \supseteq \dots \supseteq \phi : E_i \hookrightarrow \mathbb{G}$ is an ε_i -isometry, where $\varepsilon_i < \min\{\varepsilon_{i+1}/2, \frac{\varepsilon_{i+1} \cdot \delta_i}{10}\}$, where $\delta_i = \text{dist}(E_i, f_{i+1})$.

Thus we need to prove Claim 3.11. Suppose that E is a subspace of codimension 1 in a finite dimensional Banach space $F \supseteq E$. Let (e_1, \dots, e_n) be a basis of E and (e_1, \dots, e_n, v) a basis in F such that $\|v\| = 1$. Let $\varepsilon > \varepsilon' > 0$ be given, where $\varepsilon' < \min\{\varepsilon/2, \frac{\varepsilon \cdot \delta}{10}\}$, $\delta = \text{dist}(v, E)$. Moreover suppose we have an ε' -isometry $\phi : E \hookrightarrow \mathbb{G}$.

For any $\gamma > 0$ and $i \leq n$ we can find $g_i \in G \subseteq \mathbb{G}$ such that $\|g_i - \phi(e_i)\| < \gamma$. Moreover, there exists some m such that $g_i \in X_m \subseteq G$ for all i (recall again that G is a direct limit of some sequence $(X_i)_i$ from \mathcal{K}). We may even assume that each g_i is a rational linear combination of elements of the specified basis of X_m . It is clear that if we take $\gamma > 0$ small enough then the linear map $\phi' : E \hookrightarrow X_m$ determined by sending e_i to g_i is an ε'' -isometry for some $\varepsilon'' < \min\{\varepsilon/2, \frac{\varepsilon \cdot \delta}{10}\}$. We shall also later need that $\gamma \leq \varepsilon'$.

Take now $R > 0$ large enough to be specified later and $\varepsilon' > \eta > 0$ and find some finite η -net $(z_j)_j$ in $R \cdot B_E$, the compact ball of radius R in E . We may assume that each z_j from the net is a rational linear combination of the basis elements $\{e_1, \dots, e_n\}$. Consider now an (abstract) extension Z of X_m generated by X_m and one additional vector w of norm 1. The norm on X_m (which is a restriction of the norm on \mathbb{G} to this subspace) is an extension of some rational partial norm $\|\cdot\|_A$, where A is a finite set of rational linear combinations of the basis of X_m . Extend A to \bar{A} so that it contains $\phi'(z_j)$ for every j and w . Note that each z_j is a rational linear combination of basis elements $\{e_1, \dots, e_n\}$ and $\phi'(e_i)$ is a linear combination of basis elements of X_m , thus $\phi'(z_j)$ is also a rational linear combination of basis elements in X_m . Extend the partial rational norm $\|\cdot\|_A$ on A to a partial rational norm $\|\cdot\|_{\bar{A}}$ on \bar{A} so that for every j we have

$$|\|z_j - v\| - \|\phi'(z_j) - w\|_{\bar{A}}| < \varepsilon''.$$

That can be done as in the proof of Claim 2.4. We consider the norm $\|\cdot\|_{\bar{Z}}$ on Z that extends the partial norm $\|\cdot\|_{\bar{A}}$; it coincides with the X_m -norm on the subspace X_m . By Fact 3.9 this ‘abstract’ extension Z is realized in $G \subseteq \mathbb{G}$, thus we may suppose that actually w is an element of \mathbb{G} .

We now claim that the extension $\bar{\phi} \supseteq \phi : F \hookrightarrow \mathbb{G}$ determined by sending v to w is as desired. It suffices to check that for any $y \in E$ we have

$$|||v - y|| - ||w - \phi(y)||| < \varepsilon \cdot ||v - y||.$$

Suppose at first that $y \in R \cdot B_E$. Then we pick some z_j from the η -net such that $||y - z_j|| < \eta$. Then we have

$$\begin{aligned} & |||v - y|| - ||w - \phi(y)||| \leq \\ & |||v - z_j|| - ||w - \phi'(z_j)||| + ||y - z_j|| + ||\phi(y) - \phi(z_j)|| + ||\phi(z_j) - \phi'(z_j)||| < \\ & \varepsilon'' + \eta + \varepsilon' \cdot \eta + \gamma \cdot ||z_j|| < \varepsilon'' + \eta + \varepsilon' \cdot \eta + \gamma \cdot \eta + \gamma \cdot ||y|| < \\ & \varepsilon'' + \eta + \varepsilon' \cdot \eta + \varepsilon' \cdot \eta + \varepsilon' \cdot (1 + ||v - y||) < \varepsilon'' + \varepsilon'' + \varepsilon'' + \varepsilon'' + \varepsilon'' \cdot ||v - y|| < \\ & \frac{\varepsilon \cdot \text{dist}(v, E)}{2} + \frac{\varepsilon \cdot ||v - y||}{2} < \varepsilon \cdot ||v - y||. \end{aligned}$$

Suppose now that $y \notin R \cdot B_E$. Then we have

$$\begin{aligned} & |||v - y|| - ||w - \phi(y)||| \leq \\ & |||v|| + ||w|| + |||y|| - ||\phi(y)||| \leq 2 + \varepsilon' \cdot ||y|| \leq 3 + \varepsilon' \cdot ||v - y|| \leq \\ & \frac{3}{R-1} \cdot ||v - y|| + \varepsilon' \cdot ||v - y|| = \left(\frac{3}{R-1} + \varepsilon'\right) \cdot ||v - y||. \end{aligned}$$

Clearly, if R is large enough, $\frac{3}{R-1} + \varepsilon'$ is less than ε . \square

Knowing the construction of the Gurarij space which is in the similar vein as the construction of the Urysohn space we will be rather easily able to transfer the results about universal structures on the Urysohn space to analogous results on the Gurarij space, i.e. Theorems 3.3 and 3.4. The proofs will be sketchy as it is a repetition of very similar arguments to those from the section on the Urysohn space.

Proof of Theorem 3.3. We shall define an appropriate Fraïssé class. Before doing so, analogously as in Definition 3.5 and Fact 3.6 we define a partial seminorm and show to extend it.

Let X be a normed space and $A \subseteq X$ subset such that $\text{span}\{A\} = X$. Let $S_A : A \rightarrow \mathbb{R}_0^+$ be a partial 1-Lipschitz seminorm.

Then we can consider as in Fact 3.6 the greatest extension of S_A to $S_X : X \rightarrow \mathbb{R}_0^+$ as follows: for any $x \in X$ we set

$$S_X(x) = \inf\{|\alpha_1| \cdot S_A(x_1) + \dots + |\alpha_n| \cdot S_A(x_n) :$$

$$x_1, \dots, x_n \in A, x = \alpha_1 \cdot x_1 + \dots + \alpha_n \cdot x_n, \alpha_1 \geq 0, \dots, \alpha_n \geq 0\}.$$

Definition 3.12. \mathcal{K}_1 will be the following class of pairs of seminorms and Banach spaces. We have that $(S_X, X) \in \mathcal{K}_1$ if:

- X is a finite dimensional vector space with a specified basis (x_1, \dots, x_n)
- the norm $\|\cdot\|$ on X is of the form $\|\cdot\|_X^A$, i.e. determined by a partial norm $\|\cdot\|_A$. We again demand that A is a finite subset of X containing the basis such that each element of A is a linear combination of elements of the basis using only rational scalars, and $\|\cdot\|_A : A \rightarrow \mathbb{Q}$ is a partial norm taking values only in rationals. Moreover, we shall assume that each basis element x_i , $i \leq n$, is of norm 1.
- S_X is the (greatest) extension of some 1-Lipschitz seminorm $S_A : A \rightarrow \mathbb{Q}_0^+$ on A taking only rational values.

We shall again consider only proper linear embeddings, i.e. those linear isometric embeddings between elements of \mathcal{K}_1 that send elements of the basis to elements of basis, that are moreover 0-morphisms, i.e. preserve the seminorms.

The verification that \mathcal{K}_1 is a Fraïssé class is essentially the same as in Fact 3.8 plus some arguments from the proof of Theorem 2.1 on the universal subset of the Urysohn space. The Fraïssé limit (S, G) is a direct limit of some pairs $(S_1, X_1) \rightarrow (S_2, X_2) \rightarrow \dots$. Let $H \subseteq G$ be the linear subspace $\{x \in G : S(x) = 0\}$. We denote by \mathbb{G} the completion of G , by \mathbb{S} the unique extension of S and by \mathbb{H} the completion of the subspace H in \mathbb{G} . We need to check that the pair (\mathbb{S}, \mathbb{G}) satisfies the condition from the statement of Theorem 3.3. It will be again sufficient to prove the following claim.

Claim 3.13. *For every $\varepsilon > \varepsilon' > 0$, every pairs $(S_E, E) \subseteq (S_F, F)$, where*

- E, F are finite dimensional Banach spaces and E is of co-dimension 1 in F , $F = \text{span}\{E, v\}$ and $v \in F \setminus E$ such that $\|v\| = 1$
- S_E, S_F are seminorms where S_F extends S_E
- $\varepsilon' < \min\{\varepsilon/2, \frac{\varepsilon\delta}{10}\}$, where $\delta = \text{dist}(v, E) = \inf\{\|v - x\| : x \in E\}$

we have that any ε' -morphism $\phi : (S_E, E) \hookrightarrow (\mathbb{S}, \mathbb{G})$ can be extended to $\bar{\phi} \supseteq \phi : (S_F, F) \hookrightarrow (\mathbb{S}, \mathbb{G})$ such that $\bar{\phi}$ is an ε -morphism.

That is essentially the same as the proof of Claim 3.11. We again define the mapping ϕ' that goes from E to some $X_m \subseteq G \subseteq \mathbb{G}$ and sends basis elements of E to rational linear combinations of basis elements of X_m . Then we again find an appropriate η -net in a large enough ball $R \cdot B_E$. When defining the abstract extension $Z = \text{span}\{X_m, w\}$ of X_m , the only difference is that besides the norm we also have to define the seminorm S on this extension. We do it precisely the same as we

did for the norm.

We show how the ‘homogeneity condition’ from the statement of the theorem implies the universality. We do it analogously as in the proof of universality of the Gurarij space in [10].

We need the following Lemma which is analogous to Lemma 2.1 in [10].

Lemma 3.14. *Let (S_X, X) and (S_Y, Y) be pairs of finite dimensional Banach spaces together with seminorms and let $\phi : (S_X, X) \rightarrow (S_Y, Y)$ be an ε -morphism for some $\varepsilon \geq 0$. Then there exists a pair (S_W, W) , where W is finite dimensional, and 0-morphisms $\iota_X : (S_X, X) \rightarrow (S_W, W)$ and $\iota_Y : (S_Y, Y) \rightarrow (S_W, W)$ such that $\|\iota_Y \circ \phi - \iota_X\| < \varepsilon$.*

Suppose for a moment that the lemma has been proved. Then the rest is done like the proof of the universality of the Gurarij space in [10] with ε -isometries replaced by ε -morphisms. Let X be a separable Banach space together with a 1-Lipschitz seminorm $S_X : X \rightarrow \mathbb{R}_0^+$. Let $(X_n)_n$ be an increasing chain of finite dimensional subspaces of X such that $\overline{\bigcup_n X_n} = X$. Denote by S_n the restriction $S_X \upharpoonright X_n$. We inductively find linear embeddings $\phi_n : X_n \hookrightarrow \mathbb{G}$ so that

- $\phi_n : (S_n, X_n) \rightarrow (\mathbb{S}, \mathbb{G})$ is a $1/2^n$ -morphism,
- $\|\phi_{n+1} \upharpoonright X_n - \phi_n\| < 1/2^{n-1}$

Once this is done we take the point-wise limit $\phi : \bigcup_n X_n \hookrightarrow \mathbb{G}$. It uniquely extends to a linear isometric embedding still denoted by ϕ on X with the property that for each $x \in X$ we have $|S_X(x) - \mathbb{S}(\phi(x))| < \varepsilon$ for every $\varepsilon > 0$ thus $S_X(x) = \mathbb{S}(\phi(x))$. In particular, if S_X is a distance function from a closed subspace $Y \subseteq X$, then for each $x \in X$ we have $x \in Y$ iff $\phi(x) \in \mathbb{H}$.

Let us now find such linear embedding ϕ_n ’s. We assume that $X_1 = \{0\}$. Suppose we have found a $1/2^n$ -morphism $\phi_n : (S_n, X_n) \rightarrow (\mathbb{S}, \mathbb{G})$. Denote by $X'_n \subseteq \mathbb{G}$ the image $\phi_n(X_n)$ and by $\phi'_n : X'_n \rightarrow X_{n+1}$ the inverse ϕ_n^{-1} (composed with the inclusion $\subseteq X_n \rightarrow X_{n+1}$). ϕ'_n is also a $1/2^n$ -morphism. Using Lemma 3.14 we can find a pair (S_W, W) and 0-morphisms $\iota_n : (\mathbb{S} \upharpoonright X'_n, X'_n) \rightarrow (S_W, W)$ and $\iota_{n+1} : (S_{n+1}, X_{n+1}) \rightarrow (S_W, W)$ such that $\|\iota_{n+1} \circ \phi'_n - \iota_n\| < 1/2^n$. Then using the homogeneity property of (\mathbb{S}, \mathbb{G}) we can find a $1/2^{n+1}$ -morphism $\psi : (S_W, W) \rightarrow (\mathbb{S}, \mathbb{G})$ such that $\psi \circ \phi_n = \text{id}_{X'_n}$. The desired $1/2^{n+1}$ -morphism $\phi_{n+1} : (S_{n+1}, X_{n+1}) \rightarrow (\mathbb{S}, \mathbb{G})$ is then the composition $\psi \circ \iota_{n+1}$.

It remains to prove Lemma 3.14.

Proof of Lemma 3.14. We refer the reader to the proof of Lemma 2.1 in [10] which is formulated precisely the same as Lemma 3.14, just with ε -isometries instead of ε -morphisms.

Since being an ε -morphism means being an ε -isometry in both the norm and the seminorm we just use Lemma 2.1 from [10] twice. Going through the proof of that lemma we see that W is $X \oplus Y$ with a suitable norm $\|\cdot\|'$ and ι_X and ι_Y are the canonical embeddings. We then apply Lemma 2.1 from [10] again for the quotient spaces X_Q and Y_Q (quotiented by their respective seminorms) to obtain a suitable norm $\|\cdot\|''$ on $X_Q \oplus Y_Q$. The desired seminorm S_W is then the composition of projection from $X \oplus Y$ to $X_Q \oplus Y_Q$ with the norm $\|\cdot\|''$. \square

\square

Proof of Theorem 3.4. Let us start by defining the appropriate Fraïssé class.

Definition 3.15. \mathcal{K}_2 will be the class of triples of seminorms, projections and Banach spaces. We have that $(S_X, p_X, X) \in \mathcal{K}_2$ if:

- X is a finite dimensional vector space with a specified basis (x_1, \dots, x_n) ,
- the norm $\|\cdot\|$ on X is of the form $\|\cdot\|_X^A$, i.e. determined by a partial norm $\|\cdot\|_A$. We again demand that A is a finite subset of X containing the basis such that each element of A is a linear combination of elements of the basis using only rational scalars, and $\|\cdot\|_A : A \rightarrow \mathbb{Q}$ is a partial norm taking values only in rationals. Moreover, we shall assume that each basis element x_i , $i \leq n$, is of norm 1.
- S_X is again the (greatest) extension of some 1-Lipschitz seminorm $S_A : A \rightarrow \mathbb{Q}_0^+$ on A taking only rational values.
- X is equipped with a norm one projection p that is allowed to send basis elements only to rational linear combinations of basis elements and moreover, $Y_{S_X} = \{x \in X : S_X(x) = 0\} = p_X(X) = \{x \in X : p_X(x) = x\}$; i.e. S_X and p_X are compatible.

We shall again consider only proper linear embeddings, i.e. those linear isometric embeddings between elements of \mathcal{K}_2 that send elements of the basis to elements of basis, that are moreover 0-morphisms and commute with projections. So allowed embedding between (S_X, p_X, X) and (S_Y, p_Y, Y) is a 0-morphism ϕ that sends specified basis elements of X to specified basis elements of Y and for every $x \in X$, $p_Y \circ \phi(x) = \phi \circ p_X(x)$.

The verification that it is indeed a Fraïssé class is again based on the same arguments as in the proofs of Fact 3.8 and Theorem 2.7. The

Fraïssé limit G is again a direct limit of some spaces $X_1 \rightarrow X_2 \rightarrow \dots$ that are equipped with the seminorms s_1, s_2, \dots and projections p_1, p_2, \dots that also extend to the limit and then to the completion \mathbb{G} . We shall denote this limit projection P and its range by \mathbb{H} . The limit seminorm is equal to the seminorm $S_{\mathbb{G}}^{\mathbb{H}}$. In order to check the condition from the statement of Theorem 3.4 it is again sufficient to prove the following claim.

Claim 3.16. *For every $\varepsilon > \varepsilon' > 0$, every finite dimensional Banach spaces $E \subseteq F$, where*

- *E is equipped with a seminorm S_E and a compatible norm one projection p_E that both extend to F*
- *E is of co-dimension 1 in F , $F = \text{span}\{E, v\}$ and $v \in F \setminus E$ such that $\|v\| = 1$*
- *$\varepsilon' < \min\{\varepsilon/2, \frac{\varepsilon\delta}{10}\}$, where $\delta = \text{dist}(v, E) = \inf\{\|v - x\| : x \in E\}$*

we have that any ε' -morphism $\phi : (S_E, E) \hookrightarrow (S_{\mathbb{G}}^{\mathbb{H}}, \mathbb{G})$ with the property that for every $x \in E$ we have $P \circ \phi(x) = \phi \circ p(x)$, can be extended to $\bar{\phi} \supseteq \phi : (S_F, F) \hookrightarrow (S_{\mathbb{G}}^{\mathbb{H}}, \mathbb{G})$ such that $\bar{\phi}$ is an ε -morphism with the analogous property.

That is again essentially the same as the proofs of Claim 3.11 and then Claim 3.13. We define the mapping ϕ' that goes from E to some X_m and sends basis elements of E to rational linear combinations of basis elements of X_m . Then we again find an appropriate η -net in a large enough ball $R \cdot B_E$ which we may suppose contains $\phi'(e_i)$ for every $i \leq n = \dim(E)$. When defining the abstract extension $Z = \text{span}\{X_m, w\}$ of X_m , we only additionally specify to where w projects. It is analogous as in the proof of Claim 2.9.

The universality and uniqueness of (\mathbb{G}, P) is then again a standard argument. Let us only mention that when proving the universality in the same way as in Theorem 3.3 or in paper [10], we need the following lemma which is an analog of Lemma 3.14, resp. Lemma 2.1 from [10].

Lemma 3.17. *Let (S_X, p_X, X) and (S_Y, p_Y, Y) be triples consisting of finite dimensional Banach spaces equipped with seminorms and compatible norm one projections. Let $\phi : (S_X, X) \rightarrow (S_Y, Y)$ be an ε -morphism, for some $\varepsilon > 0$, with the property that $p_Y \circ \phi = \phi \circ p_X$. Then there exists a finite dimensional W with a seminorm S_W and a compatible norm one projection p_W , and 0-morphisms $\iota_X : (S_X, X) \rightarrow (S_W, W)$, resp. $\iota_Y : (S_Y, Y) \rightarrow (S_W, W)$ with the property that for $Q \in \{X, Y\}$ we have*

$$p_W \circ \iota_Q = \iota_Q \circ p_Q$$

and moreover we have

$$\|\iota_Y \circ \phi - \iota_X\| < \varepsilon.$$

We just copy the proof of Lemma 3.14, where $W = X \oplus Y$. We only additionally define a projection p_W on $W = X \oplus Y$ which is the sum $p_X \oplus p_Y : X \oplus Y \rightarrow p_X(X) \oplus p_Y(Y)$. \square

3.1. Final remarks and problems. Consider the Urysohn space together with a universal closed subset $C \subseteq \mathbb{U}$ as guaranteed by Theorem 2.1. Can this universal closed subset be lifted to a universal subspace of \mathbb{H} ? Resp. is $F(C) \subseteq \mathbb{H}$ a universal subspace? Using just Theorem 2.22 as in the proof of Theorem 2.23 does not seem to work. Maybe a modification of Theorem 2.22 is needed.

The same question applies to the universal retraction on \mathbb{U} . Is the unique linear extension $\hat{R} : \mathbb{H} \rightarrow F(F_{\mathbb{U}})$ of the universal retraction R a universal projection on the Holmes onto a universal complemented subspace $F(F_{\mathbb{U}})$? Again, the approach from the proof of Theorem 2.23 that uses Theorem 2.22 does not seem to work directly.

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